

Continuous Quantum Hypothesis Testing

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I propose a general quantum hypothesis testing theory that enables one to test hypotheses about any aspect of a physical system, including its dynamics, based on a series of observations. For example, the hypotheses can be about the presence of a weak classical signal continuously coupled to a quantum sensor, or about competing quantum or classical models of the dynamics of a system. This generalization makes the theory useful for quantum detection and experimental tests of quantum mechanics in general. In the case of continuous measurements, the theory is significantly simplified to produce compact formulae for the likelihood ratio, the central quantity in statistical hypothesis testing. The likelihood ratio can then be computed efficiently in many cases of interest. Two potential applications of the theory, namely quantum detection of a classical stochastic waveform and test of harmonic-oscillator energy quantization, are discussed.

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Testing hypotheses about a physical system by observation is a fundamental endeavor in scientific research. Observations are often indirect, noisy, and limited; to choose the best model of a system among potential candidates, statistical inference is the most logical way [1, 2] and has been extensively employed in diverse fields of science and engineering.

Many important quantum mechanics experiments, such as tests of quantum mechanics [3, 4], quantum detection of weak forces or magnetic fields [5], and quantum target detection [6, 7], are examples of hypothesis testing. To test quantum nonlocality, for instance, one should compare the quantum model with the best classical model; Bell's inequality and its variations, which impose general bounds on observations of local-hidden-variable systems, have been widely used in this regard [3, 4]. The analyses of experimental data in many such tests have nonetheless been criticized by Peres [8]: The statistical averages in all these inequalities can never be measured exactly in a finite number of trials. One should use statistical inference to account for the uncertainties and provide an operational meaning to the data.

Another important recent development in quantum physics is the experimental demonstration of quantum behavior in increasingly macroscopic systems, such as mechanical oscillators [9, 10] and microwave resonators [11]. To test the quantization of the oscillator energy [10, 11], for example, the use of quantum filtering theory has been proposed to process the data [12], but testing quantum behavior by assuming quantum mechanics can be criticized as begging the question. An ingenious proposal by Clerk *et al.* considers the third moment of energy as a test of energy quantization [13]. Like the correlations in Bell's inequality, however, the third moment is a sta-

tistical average and cannot be measured exactly in finite time. Again, statistical inference should be used to test the quantum behavior of a system rigorously, especially when the measurements are weak and noisy. The good news here is that the error probabilities for hypothesis testing should decrease exponentially with the number of measurements when the number is large [2], so one can always compensate for a weak signal-to-noise ratio by increasing the number of trials.

Quantum hypothesis testing was first studied by Holevo [14] and Yuen *et al.* [15]. Since then, researchers have focused on the use of statistical hypothesis testing techniques for initial quantum state discrimination [6, 16]. Here I propose a more general quantum theory of hypothesis testing for *model* discrimination, allowing the hypotheses to be not just about the initial state but also about the dynamics of the system under a series of observations. This generalization makes the theory applicable to virtually any hypothesis testing problem that involves quantum mechanics, including tests of quantum dynamics [4] and quantum waveform detection [5].

In the case of continuous measurements with Gaussian or Poissonian noise, the theory is significantly simplified to produce compact formulae for the likelihood ratio, the central quantity in statistical hypothesis testing. The formulae enable one to compute the ratio efficiently in many cases of interest and should be useful for numerical approximations in general. Notable prior work on continuous quantum hypothesis testing is reported in Refs. [17, 18], which study state discrimination or parameter estimation only and have not derived the general likelihood-ratio formulae proposed here.

To illustrate the theory, I discuss two potential applications, namely quantum detection of a classical stochastic waveform and test of harmonic-oscillator energy quantization. Waveform detection is a basic operation in future quantum sensing applications, such as gravitational-wave detection, optomechanical force detection, and atomic

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magnetometry [5]. Tests of energy quantization, on the other hand, have become increasingly popular in experimental physics due to the rapid recent progress in device fabrication technologies [9–11]. Besides these two applications, the theory is expected to find wide use in quantum information processing and quantum physics in general, whenever new claims about a quantum system need to be tested rigorously.

Statistical hypothesis testing entails the comparison of observation probabilities conditioned on different hypotheses [1, 2]. To test two hypotheses labeled \mathcal{H}_0 and \mathcal{H}_1 using an observation record Y , the observer splits the observation space into two parts Z_0 and Z_1 ; when Y falls in Z_0 , the observer chooses \mathcal{H}_0 , and when Y falls in Z_1 , the observer chooses \mathcal{H}_1 . The error probabilities are then $P_{01} \equiv \int_{Z_0} dY P(Y|\mathcal{H}_1)$ and $P_{10} \equiv \int_{Z_1} dY P(Y|\mathcal{H}_0)$. All binary hypothesis testing protocols involve the computation of the likelihood ratio, defined as

$$\Lambda \equiv \frac{P(Y|\mathcal{H}_1)}{P(Y|\mathcal{H}_0)}. \quad (1)$$

The ratio is then compared against a threshold γ that depends on the protocol; one decides on \mathcal{H}_1 if $\Lambda \geq \gamma$ and \mathcal{H}_0 if $\Lambda < \gamma$. For example, the Neyman-Pearson criterion minimizes P_{01} under a constraint on P_{10} , while the Bayes criterion minimizes $aP_{01} + bP_{10}$, a and b being arbitrary positive numbers. For multiple independent trials, the final likelihood ratio is simply the product of the ratios.

In most cases, the error probabilities are difficult to calculate analytically and only bounds, such as the Chernoff upper bound [2], may be available, but the likelihood ratio can be used to update the posterior hypothesis probabilities from prior probabilities P_0 and P_1 via $P(\mathcal{H}_1|Y) = P_1\Lambda/(P_1\Lambda + P_0)$ and $P(\mathcal{H}_0|Y) = P_0/(P_1\Lambda + P_0)$ and therefore quantifies the strength of evidence for \mathcal{H}_1 against \mathcal{H}_0 given Y [1]. Generalization to multiple hypotheses beyond two is also possible by computing multiple likelihood ratios or the posterior probabilities $P(\mathcal{H}_j|Y)$ [2].

Consider now two hypotheses about a system under a sequence of measurements, with results $Y \equiv (\delta y_1, \dots, \delta y_M)$. For generality, I use quantum theory to derive $P(Y|\mathcal{H}_j)$ for both hypotheses, but note that a classical model can always be expressed mathematically as a special case of a quantum model. The observation probability distribution is [19]

$$P(Y|\mathcal{H}_j) = \text{tr} [\mathcal{J}_j(\delta y_M, t_M) \mathcal{K}_j(t_M) \dots \mathcal{J}_j(\delta y_1, t_1) \mathcal{K}_j(t_1) \rho_j(t_0)], \quad (2)$$

where $\rho_j(t_0)$ is the initial density operator at time t_0 , $\mathcal{K}_j(t_m)$ is the completely positive map that models the system dynamics from time t_{m-1} to t_m , $\mathcal{J}_j(\delta y_m, t_m)$ is the completely positive map that models the measurement at time t_m , and the subscripts j for $\rho_j(t_0)$, \mathcal{K}_j , and \mathcal{J}_j denote the assumption of \mathcal{H}_j for these quantities.

To proceed, let $t_m = t_0 + m\delta t$ and assume the following

Kraus form of \mathcal{J}_j for Gaussian measurements [19, 20]:

$$\begin{aligned} \mathcal{J}_j(\delta y, t) \rho &= \int_{-\infty}^{\infty} d(\delta z) \frac{1}{\sqrt{2\pi S_j \delta t}} \exp \left[-\frac{(\delta y - \delta z)^2}{2S_j \delta t} \right] \\ &\quad \times M_j(\delta z, t) \rho M_j^\dagger(\delta z, t), \\ M_j(\delta z, t) &\equiv \frac{1}{(2\pi Q_j \delta t)^{1/4}} \exp \left(-\frac{\delta z^2}{4Q_j \delta t} \right) \\ &\quad \times \left[1 + \frac{\delta z}{2Q_j} c_j - \frac{\delta t}{8Q_j} c_j^\dagger c_j + o(\delta t) \right], \end{aligned} \quad (3)$$

where Q_j is the noise variance of the inherent quantum-limited measurement, S_j is the excess noise variance, c_j is a quantum operator depending on the measurement, and $o(\delta t)$ denotes terms asymptotically smaller than δt . The map becomes

$$\begin{aligned} \mathcal{J}_j(\delta y, t) \rho &= \tilde{P}(\delta y) \left[\rho + \frac{\delta y}{2R} (c_j \rho + \rho c_j^\dagger) \right. \\ &\quad \left. + \frac{\delta t}{8Q_j} (2c_j \rho c_j^\dagger - c_j^\dagger c_j \rho - \rho c_j^\dagger c_j) + o(\delta t) \right], \\ \tilde{P}(\delta y) &\equiv \frac{1}{\sqrt{2\pi R \delta t}} \exp \left(-\frac{\delta y^2}{2R \delta t} \right), \quad R \equiv Q_j + S_j. \end{aligned} \quad (4)$$

I assume that the total noise variance R is independent of the hypothesis to focus on tests of hidden models rather than the observation noise levels. $\tilde{P}(\delta y)$ then factors out of both the numerator and denominator of the likelihood ratio and cancels itself.

Taking the continuous time limit using Itô calculus with $\delta y^2 = R \delta t + o(\delta t)$, the likelihood ratio becomes $\Lambda = \text{tr} f_1 / \text{tr} f_0$, with f_j obeying the following stochastic differential equation:

$$\begin{aligned} df_j &= dt \mathcal{L}_j f_j + \frac{dy}{2R} (c_j f_j + f_j c_j^\dagger) \\ &\quad + \frac{dt}{8Q_j} (2c_j f_j c_j^\dagger - c_j^\dagger c_j f_j - f_j c_j^\dagger c_j) \end{aligned} \quad (5)$$

and \mathcal{L}_j being the Lindblad generator originating from \mathcal{K}_j . Equation (5) has the exact same mathematical form as the linear Belavkin equation for an unnormalized filtering density operator [21], but beware that f_j represents the state of the system only if \mathcal{H}_j is true; I call f_j an assumptive state.

To put Λ in a form more amenable to numerics, consider the stochastic differential equation for $\text{tr} f_j$:

$$d \text{tr} f_j = \text{tr} df_j = \frac{dy}{2R} \text{tr} (c_j f_j + f_j c_j^\dagger) = \frac{dy}{R} \mu_j \text{tr} f_j, \quad (6)$$

where

$$\mu_j \equiv \frac{1}{\text{tr} f_j} \text{tr} \left(\frac{c_j + c_j^\dagger}{2} f_j \right) \quad (7)$$

is an assumptive estimate; it is the posterior mean of the observable $(c_j + c_j^\dagger)/2$ only if \mathcal{H}_j is true. The form

of Eq. (6) suggests that it can be solved by taking the logarithm of $\text{tr } f_j$, i.e., $d \ln \text{tr } f_j = (d \text{tr } f_j) / \text{tr } f_j - (d \text{tr } f_j)^2 / 2(\text{tr } f_j)^2 = dy \mu_j / R - dt \mu_j^2 / 2R$, resulting in

$$\ln \text{tr } f_j(T) = \int_{t_0}^T \frac{dy}{R} \mu_j - \int_{t_0}^T \frac{dt}{2R} \mu_j^2, \quad (8)$$

with the dy integral being an Itô integral. Λ becomes

$$\Lambda(T) = \exp \left[\int_{t_0}^T \frac{dy}{R} (\mu_1 - \mu_0) - \int_{t_0}^T \frac{dt}{2R} (\mu_1^2 - \mu_0^2) \right]. \quad (9)$$

This compact formula for the likelihood ratio is the quantum generalization of a similar result by Duncan and Kailath in classical detection theory [22]. Generalization to the case of vectorial observations with noise covariance matrix R is trivial; the result is simply Eq. (9) with $dy \mu_j / R$ replaced by $dy^\top R^{-1} \mu_j$ and μ_j^2 / R by $\mu_j^\top R^{-1} \mu_j$.

For continuous measurements with Poissonian noise, a formula for Λ can be derived similarly [23]:

$$\Lambda(T) = \exp \left[\int_{t_0}^T dy \ln \frac{\mu_1}{\mu_0} - \int_{t_0}^T dt (\mu_1 - \mu_0) \right], \quad (10)$$

$$\mu_j \equiv \frac{1}{\text{tr } f_j} \text{tr}(\eta_j c_j^\dagger c_j f_j), \quad (11)$$

$$df_j = dt \mathcal{L}_j f_j + (dy - \alpha dt) \left(\frac{\eta_j}{\alpha} c_j f_j c_j^\dagger - f_j \right) + \frac{dt}{2} \left(2c_j f_j c_j^\dagger - c_j^\dagger c_j f_j - f_j c_j^\dagger c_j \right), \quad (12)$$

where $0 < \eta_j \leq 1$ is the quantum efficiency, α can be any positive number, and Eqs. (11) and (12) form a quantum filter for Poissonian observations [20, 21]. Equation (10) generalizes a similar classical result by Snyder [24].

Equations (9) and (10) show that continuous hypothesis testing can be done simply by comparing how the observation process is correlated with the observable estimated by each hypothesis, as schematically depicted in Fig. 1.

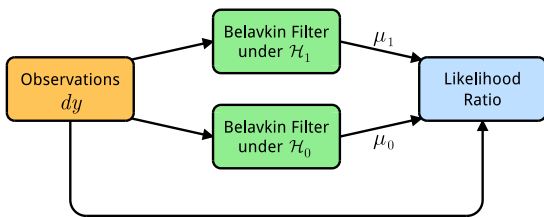


FIG. 1: (Color online). Structure of the likelihood-ratio formulae given by Eqs. (9) and (10).

Since Eqs. (5) and (7) or Eqs. (11) and (12) have the same form as Belavkin filters, one can leverage established quantum filtering techniques to update the estimates and the likelihood ratio continuously with incoming observations. If f_j has a Wigner function that remains Gaussian in time, the problem has an equivalent

classical linear Gaussian model [19–21] conditioned on each hypothesis, and μ_j can be computed efficiently using the Kalman-Bucy filter, which gives the mean vector and covariance matrix of the Wigner function. The classical model also enables one to use existing formulae of Chernoff bounds for classical waveform detection [25] to bound the error probabilities. It remains a technical challenge to compute the quantum filter for problems without a Gaussian phase-space representation beyond few-level systems, but the quantum trajectory method should help cut the required computational resources by employing an ensemble of wavefunctions instead of a density matrix [19, 26]. Error bounds for such nonclassical problems also remain an important open problem.

As an illustration of the theory, consider the detection of a weak classical stochastic signal, such as a gravitational wave or a magnetic field, using a quantum sensor [5], with \mathcal{H}_1 hypothesizing the presence of the signal and \mathcal{H}_0 its absence. Let x be a vector of the state variables for the classical signal. One way to account for the dynamics of x is to use the hybrid density operator formalism, which includes x as auxiliary degrees of freedom in the system [18, 20, 27]. The initial assumptive state $f_1(t_0)$ becomes $\rho(t_0)P(x, t_0)$, with $\rho(t_0)$ being the initial density operator for the quantum sensor and $P(x, t_0)$ the initial probability density of x . Equation (5) for f_1 becomes

$$df_1 = dt \mathcal{L}_1(x) f_1 + \frac{dy}{2R} (c f_1 + f_1 c^\dagger) + \frac{dt}{8Q} (2c f_1 c^\dagger - c^\dagger c f_1 - f_1 c^\dagger c), \quad (13)$$

with $\mu_1 = \int dx \text{tr}[(c + c^\dagger) f_1 / 2] / \int dx \text{tr } f_1$. $\mathcal{L}_1(x)$ should include the Lindblad generator for the quantum sensor, the coupling of x to the quantum sensor via an interaction Hamiltonian, and also the forward Kolmogorov generator that models the classical dynamics of x [20]. c is an operator that depends on the actual measurement of the quantum sensor; for cavity optomechanical force detection for example, c is the cavity optical annihilation operator or can be approximated as the mechanical position operator if the intracavity optical dynamics can be adiabatically eliminated [28].

For the null hypothesis \mathcal{H}_0 , the classical degrees of freedom need not be included. $f_0(t_0)$ is then $\rho(t_0)$, Eq. (5) becomes

$$df_0 = dt \mathcal{L}_0 f_0 + \frac{dy}{2R} (c f_0 + f_0 c^\dagger) + \frac{dt}{8Q} (2c f_0 c^\dagger - c^\dagger c f_0 - f_0 c^\dagger c), \quad (14)$$

and \mathcal{L}_0 includes only the Lindblad generator for the quantum sensor. In most current cases of interest in quantum sensing, the Wigner functions for f_0 and f_1 remain approximately Gaussian [5, 20]. Kalman-Bucy filters can then be used to solve Eqs. (13) and (14) for the assumptive estimates, to be correlated with the observation process according to Eq. (9) to produce Λ , and existing formulae of Chernoff bounds for classical waveform detection

[25] can be used to bound the error probabilities. Ref. [23] contains a simple example of such calculations.

Quantum smoothing can further improve the estimation of x [20] in the event of a likely detection. Although smoothing is not needed here for the exact computation of Λ , it may be useful for improving the approximation of Λ for non-Gaussian problems when the exact estimates are too expensive to compute [29].

As a second example, consider the test of energy quantization in a harmonic oscillator. To ensure the rigor of the test, imagine a classical physicist who wishes to challenge the quantum harmonic oscillator model by proposing a competing model based on classical mechanics. To devise a good classical model, he first examines quadrature measurements of a harmonic oscillator in a thermal bath. With the harmonic time dependence on the oscillator frequency removed in an interaction picture, the assumptive state f_1 for the quantum hypothesis \mathcal{H}_1 obeys

$$\begin{aligned} df_1 = & \frac{\gamma dt}{2} [(N+1)(2af_1a^\dagger - a^\dagger af_1 - f_1a^\dagger a) \\ & + N(2a^\dagger f_1a - aa^\dagger f_1 - f_1aa^\dagger)] + \frac{dy}{2R}(cf_1 + f_1c) \\ & + \frac{dt}{8Q}(2cf_1c - c^2f_1 - f_1c^2), \end{aligned} \quad (15)$$

$$c = q \cos \theta + p \sin \theta, \quad (16)$$

where $a \equiv (q + ip)/\sqrt{2}$ is the annihilation operator, q and p are quadrature operators, c is a quadrature operator with θ held fixed for each trial to eliminate any complicating measurement backaction effect, γ is the decay rate of the oscillator, and N is a temperature-dependent parameter. This backaction-evading measurement scheme can be implemented approximately by double-sideband optical pumping in cavity optomechanics [5, 30].

An equivalent classical model for the quadrature measurements is

$$\begin{aligned} dx_1 = & -\frac{\gamma}{2}x_1dt + \sqrt{\gamma}dW_1, \quad dx_2 = -\frac{\gamma}{2}x_2dt + \sqrt{\gamma}dW_2, \\ dy = & h(x_1, x_2)dt + dV, \end{aligned} \quad (17)$$

$$h = x_1 \cos \theta + x_2 \sin \theta. \quad (18)$$

where x_1 and x_2 are classical Ornstein-Uhlenbeck processes and dW_1 , dW_2 , and dV are uncorrelated classical Wiener noises with $dW_1^2 = dW_2^2 = (N+1/2)dt$ and $dV^2 = Rdt$. One can make f_0 diagonal and embed it with a classical distribution $g_0(x_1, x_2)$ to model classical statistics; the equation for the classical assumptive state g_0 is

$$\begin{aligned} dg_0 = & \frac{\gamma dt}{2} \left[\frac{\partial}{\partial x_1}(x_1g_0) + \frac{\partial}{\partial x_2}(x_2g_0) \right. \\ & \left. + \left(N + \frac{1}{2} \right) \left(\frac{\partial^2 g_0}{\partial x_1^2} + \frac{\partial^2 g_0}{\partial x_2^2} \right) \right] + \frac{dy}{R}hg_0. \end{aligned} \quad (19)$$

which is a classical Duncan-Mortensen-Zakai (DMZ) equation [31]. The assumptive estimate $\mu_0 = \int dx_1 dx_2 h g_0 / \int dx_1 dx_2 g_0$ should be identical to the quantum one, as can be seen by transforming f_1 to a Wigner function and neglecting the measurement backaction that does not affect the observations. Λ given the quadrature observations then stays at 1, confirming that the two models are indistinguishable.

In a different experiment on the same oscillator, the energy of the oscillator is measured instead. Let

$$c = \frac{q^2 + p^2}{2} = \frac{a^\dagger a + aa^\dagger}{2}, \quad (20)$$

which can be implemented approximately by dispersive optomechanical coupling in cavity optomechanics [5, 10, 12]. f_1 still obeys Eq. (15), but with c now given by Eq. (20) and different R and Q . The measurements are again backaction-evading, as the backaction noise on the oscillator phase does not affect the energy observations.

Given the prior success of the classical model, the classical physicist decides to retain Eqs. (17) and modifies only the observation as a function of x_1 and x_2 :

$$h = \frac{x_1^2 + x_2^2}{2}. \quad (21)$$

The DMZ equation given by Eq. (19), assuming continuous energy, should now produce an assumptive energy estimate different from the quantum one; it is this difference that should make the likelihood ratio increase in favor of the quantum hypothesis with more observations, if quantum mechanics is correct. Previous data analysis techniques that consider only the quantum estimate [12] fail to take into account the probability that the observations can also be explained by a continuous-energy model and are therefore insufficient to demonstrate energy quantization conclusively. The non-Gaussian nature of the problem means that bounds on the error probabilities may be difficult to compute analytically and one may have to resort to numerics, but one can also use Λ as a Bayesian statistic to quantify the strength of the evidence for one hypothesis against another [1].

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Appendix A: Supplementary Material

This document contains a derivation of the likelihood-ratio formula for continuous quantum measurements with Poissonian noise in Sec. A 1 and an example of quantum optomechanical stochastic force detection in Sec. A 2.

1. Likelihood-ratio formula for continuous Poissonian measurements

The completely positive map for a weak Poissonian measurement is given by

$$\mathcal{J}_j(\delta y)\rho = \sum_{\delta z=0,1} P(\delta y|\delta z) \left\{ \delta z c_j \rho c_j^\dagger \delta t + (1 - \delta z) \left[\rho - \frac{\delta t}{2} (c_j^\dagger c_j \rho + \rho c_j^\dagger c_j) \right] \right\}, \quad (\text{A1})$$

where $\delta y, \delta z \in \{0, 1\}$ and

$$P(\delta y|\delta z) = (1 - \delta y)(1 - \eta_j \delta z) + \eta_j \delta y \delta z \quad (\text{A2})$$

models the effect of imperfect quantum efficiency $0 < \eta_j \leq 1$. Rearranging terms [20],

$$\mathcal{J}_j(\delta y)\rho = \tilde{P}(\delta y) \left[\rho + \frac{\delta t}{2} (2c_j \rho c_j^\dagger - c_j^\dagger c_j \rho - \rho c_j^\dagger c_j) + (\delta y - \alpha \delta t) \left(\frac{\eta_j}{\alpha} c_j \rho c_j^\dagger - \rho \right) \right], \quad (\text{A3})$$

$$\tilde{P}(\delta y) \equiv (1 - \delta y)(1 - \alpha \delta t) + \delta y \alpha \delta t, \quad (\text{A4})$$

where $\tilde{P}(\delta y)$ is a reference probability distribution and α is an arbitrary positive number. This gives

$$\Lambda = \frac{\text{tr } f_1}{\text{tr } f_0}, \quad (\text{A5})$$

$$df_j = dt \mathcal{L}_j f_j + \frac{dt}{2} (2c_j f_j c_j^\dagger - c_j^\dagger c_j f_j - f_j c_j^\dagger c_j) + (dy - \alpha dt) \left(\frac{\eta_j}{\alpha} c_j f_j c_j^\dagger - f_j \right). \quad (\text{A6})$$

Equation (A6) coincides with the quantum filtering equation for an unnormalized posterior density operator f_j given Poissonian observations dy [20]. Next, consider

$$d \text{tr } f_j = \text{tr } df_j = (dy - \alpha dt) \left(\frac{\mu_j}{\alpha} - 1 \right) \text{tr } f_j, \quad (\text{A7})$$

$$\mu_j \equiv \frac{1}{\text{tr } f_j} \text{tr} \left(\eta_j c_j^\dagger c_j f_j \right), \quad (\text{A8})$$

where μ_j is the filtering estimate of the observable $\eta_j c_j^\dagger c_j$ assuming that the hypothesis \mathcal{H}_j is true. Expanding $d \ln \text{tr } f_j$ in Taylor series,

$$d \ln \text{tr } f_j = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n (\text{tr } f_j)^n} (d \text{tr } f_j)^n \quad (\text{A9})$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (dy - \alpha dt)^n \left(\frac{\mu_j}{\alpha} - 1 \right)^n. \quad (\text{A10})$$

With $(dy - \alpha dt)^n = dy^n + o(dt)$ for $n \geq 2$ and $dy^n = dy$ for Poissonian observations,

$$d \ln \text{tr } f_j = dy \ln \frac{\mu_j}{\alpha} - dt (\mu_j - \alpha), \quad (\text{A11})$$

$$\ln \text{tr } f_j(T) = \int_{t_0}^T dy \ln \frac{\mu_j}{\alpha} - \int_{t_0}^T dt (\mu_j - \alpha), \quad (\text{A12})$$

$$\Lambda(T) = \exp \left[\int_{t_0}^T dy \ln \frac{\mu_1}{\mu_0} - \int_{t_0}^T dt (\mu_1 - \mu_0) \right]. \quad (\text{A13})$$

2. Quantum optomechanical detection of a Gaussian stochastic force

Let $F = Cx$ be a classical force acting on a moving mirror with position operator q and momentum p , and x be a vectorial classical Gaussian stochastic process x described by the Ito equation

$$dx = Axdx + dW, \quad dWdW^\top = Bdx. \quad (\text{A14})$$

The mirror is assumed to be a harmonic oscillator with mass m and frequency ω and part of an optical cavity pumped by a near-resonant continuous-wave laser. The phase quadrature of the cavity output is measured continuously by homodyne detection, with an observation process given by dy . For simplicity, I assume that the optical intracavity dynamics can be adiabatically eliminated, the phase modulation by the mirror motion is much smaller than $\pi/2$ radians, such that the homodyne detection is effectively measuring the mirror position, and there is no excess decoherence. Under hypothesis \mathcal{H}_1 , the force is present and the quantum filtering equation for the unnormalized hybrid density operator $f_1(x, t)$ is then given by [20]

$$df_1 = dt\mathcal{L}_1(x)f_1 + \frac{dy}{2R}(qf_1 + f_1q) + \frac{dt}{8R}(2qf_1q - q^2f_1 - f_1q^2), \quad (\text{A15})$$

$$\mathcal{L}_1(x)f_1 \equiv -\frac{i}{\hbar}[H_1(x), f_1] + \mathcal{L}_c(x)f_1, \quad (\text{A16})$$

$$H_1(x) \equiv \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2 - qCx, \quad (\text{A17})$$

$$\mathcal{L}_c(x)f_1 \equiv -\sum_{\mu} \frac{\partial}{\partial x_{\mu}} [(Ax)_{\mu}f_1] + \frac{1}{2} \sum_{\mu, \nu} \frac{\partial^2}{\partial x_{\mu} \partial x_{\nu}} (B_{\mu\nu}f_1), \quad (\text{A18})$$

where R is the measurement noise variance that depends on the laser intensity and the cavity properties and $\mathcal{L}_c(x)$ is the forward Kolmogorov generator for the classical process x . Under the null hypothesis \mathcal{H}_0 , the force is absent and the filtering equation for the oscillator density operator f_0 is

$$df_0 = dt\mathcal{L}_0f_0 + \frac{dy}{2R}(qf_0 + f_0q) + \frac{dt}{8R}(2qf_0q - q^2f_0 - f_0q^2), \quad (\text{A19})$$

$$\mathcal{L}_0f_0 \equiv -\frac{i}{\hbar}[H_0, f_0], \quad (\text{A20})$$

$$H_0 \equiv \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2. \quad (\text{A21})$$

These filtering equations can be transformed to equations for the Wigner functions of f_j :

$$g_1(q, p, x, t) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} du \langle q - u/2 | f_1(x, t) | q + u/2 \rangle \exp(ipu/\hbar), \quad (\text{A22})$$

$$g_0(q, p, t) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} du \langle q - u/2 | f_0(t) | q + u/2 \rangle \exp(ipu/\hbar), \quad (\text{A23})$$

$$dg_j = dt\mathcal{L}'_jg_j + \frac{dy}{R}qg_j, \quad (\text{A24})$$

$$\mathcal{L}'_1g_1 \equiv \mathcal{L}_c(x)g_1 + dt \left[-\frac{p}{m} \frac{\partial g_1}{\partial q} + (m\omega_m^2q - Cx) \frac{\partial g_1}{\partial p} + \frac{\hbar^2}{8R} \frac{\partial^2 g_1}{\partial p^2} \right], \quad (\text{A25})$$

$$\mathcal{L}'_0g_0 \equiv -\frac{p}{m} \frac{\partial g_0}{\partial q} + m\omega^2q \frac{\partial g_0}{\partial p} + \frac{\hbar^2}{8R} \frac{\partial^2 g_0}{\partial p^2}, \quad (\text{A26})$$

where q and p are now phase-space variables, p is seen to suffer from measurement-back-action-induced diffusion, and \mathcal{L}'_1 has the form of a forward Kolmogorov generator for a new Gaussian process $z = (q, p, x^\top)^\top$:

$$\mathcal{L}'_1g_1(z, t) = -\sum_{\mu} \frac{\partial}{\partial z_{\mu}} [(J_1z)_{\mu}g_1] + \frac{1}{2} \sum_{\mu, \nu} \frac{\partial^2}{\partial z_{\mu} \partial z_{\nu}} (S_{1\mu\nu}g_1), \quad (\text{A27})$$

$$J_1 \equiv \left(\begin{array}{cc|c} 0 & 1/m & \mathbf{0} \\ -m\omega^2 & 0 & C \\ \hline \mathbf{0} & \mathbf{0} & A \end{array} \right), \quad S_1 \equiv \left(\begin{array}{cc|c} 0 & 0 & \mathbf{0} \\ 0 & \hbar^2/4R & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & B \end{array} \right), \quad (\text{A28})$$

with $\mathbf{0}$ denoting zero matrices. Similarly, under \mathcal{H}_0 , we have $z = (q, p)^\top$ and

$$\mathcal{L}'_0 g_0(z, t) = - \sum_{\mu} \frac{\partial}{\partial z_{\mu}} [(J_0 z)_{\mu} g_0] + \frac{1}{2} \sum_{\mu, \nu} \frac{\partial^2}{\partial z_{\mu} \partial z_{\nu}} (S_{0\mu\nu} g_0), \quad (\text{A29})$$

$$J_0 \equiv \begin{pmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{pmatrix}, \quad S_0 \equiv \begin{pmatrix} 0 & 0 \\ 0 & \hbar^2/4R \end{pmatrix}. \quad (\text{A30})$$

The Gaussian statistics mean that we can use Kalman-Bucy filters to compute the filtering estimates of the mirror position μ_j given dy [20]:

$$dz'_j = J_j z'_j dt + \Gamma_j (dy - K_j z'_j dt), \quad K_j \equiv (1, 0, \dots, 0), \quad (\text{A31})$$

$$\Gamma_j \equiv \Sigma_j K_j^\top R^{-1}, \quad (\text{A32})$$

$$\frac{d\Sigma_j}{dt} = J_j \Sigma_j + \Sigma_j J_j^\top - \Sigma_j K_j^\top R^{-1} K_j \Sigma_j^\top + S_j, \quad (\text{A33})$$

$$\mu_j = K_j z'_j, \quad (\text{A34})$$

and the likelihood ratio becomes

$$\Lambda(T) = \exp \left[\int_{t_0}^T \frac{dy}{R} (\mu_1 - \mu_0) - \int_{t_0}^T \frac{dt}{2R} (\mu_1^2 - \mu_0^2) \right]. \quad (\text{A35})$$

Given the Gaussian structure of the problem under each hypothesis, we can use known results about the Chernoff upper bounds for classical waveform estimation to bound the error probabilities [25]:

$$P_{10} \leq \exp [\mu(s) - s\gamma], \quad (\text{A36})$$

$$P_{01} \leq \exp [\mu(s) + (1 - s)\gamma], \quad (\text{A37})$$

$$\mu(s) = \frac{1}{2R} \int_{t_0}^T dt \left[(1 - s) \Sigma_{1q}(t) + s \Sigma_{0q}(t) - \tilde{\Sigma}_q(s, t) \right], \quad (\text{A38})$$

where $0 \leq s \leq 1$, γ is the threshold of the likelihood-ratio test, $\Sigma_{jq}(t)$ is the q variance component of Σ_j , which obeys Eq. (A33), and $\tilde{\Sigma}_q(s, t)$ is the variance of $\sqrt{s}q_0 + \sqrt{1-s}q_1$ for a different filtering problem, in which observations of $\sqrt{s}q_0 + \sqrt{1-s}q_1$ are made with noise variance R and q_j has the statistics of q under \mathcal{H}_j , viz.,

$$\frac{d\tilde{\Sigma}}{dt} = \tilde{J}\tilde{\Sigma} + \tilde{\Sigma}\tilde{J}^\top - \tilde{\Sigma}\tilde{K}^\top R^{-1} \tilde{K}\tilde{\Sigma}^\top + \tilde{S}, \quad (\text{A39})$$

$$\tilde{J} \equiv \left(\begin{array}{c|c} J_0 & \mathbf{0} \\ \hline \mathbf{0} & J_1 \end{array} \right), \quad \tilde{S} \equiv \left(\begin{array}{c|c} S_0 & \mathbf{0} \\ \hline \mathbf{0} & S_1 \end{array} \right), \quad \tilde{K} \equiv \left(\begin{array}{c|c} \sqrt{s} & \mathbf{0} \\ \hline \mathbf{0} & \sqrt{1-s} \end{array} \right), \quad \tilde{\Sigma}_q \equiv \tilde{K}\tilde{\Sigma}\tilde{K}^\top. \quad (\text{A40})$$

The tightest upper bounds are obtained by minimizing the bounds with respect to s . If x and therefore q are stationary, Σ_j and $\tilde{\Sigma}$ will converge to steady states in the long-time limit, and the Chernoff bounds will decay exponentially with time.
